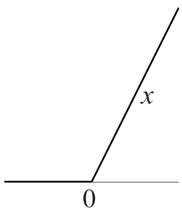
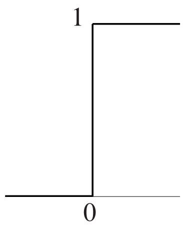
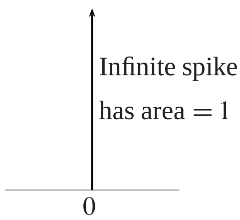
Summary: Six Functions, Six Rules, Six Theorems

Integrals	**Six Functions**	*Derivatives*
$x^{n+1}/(n+1),\ n \neq -1$	$\boldsymbol{x^n}$	nx^{n-1}
$-\cos x$	$\boldsymbol{\sin x}$	$\cos x$
$\sin x$	$\boldsymbol{\cos x}$	$-\sin x$
e^{cx}/c	$\boldsymbol{e^{cx}}$	ce^{cx}
$x \ln x - x$	$\boldsymbol{\ln x}$	$1/x$
Ramp function	**Step function**	**Delta function**
		

Six Rules of Differential Calculus

1. The derivative of $\boldsymbol{af(x)+bg(x)}$ is $a\dfrac{df}{dx}+b\dfrac{dg}{dx}$ **Sum**

2. The derivative of $\boldsymbol{f(x)g(x)}$ is $f(x)\dfrac{dg}{dx}+g(x)\dfrac{df}{dx}$ **Product**

3. The derivative of $\dfrac{\boldsymbol{f(x)}}{\boldsymbol{g(x)}}$ is $\left(g\dfrac{df}{dx}-f\dfrac{dg}{dx}\right)\Big/g^2$ **Quotient**

4. The derivative of $\boldsymbol{f(g(x))}$ is $\dfrac{df}{dy}\dfrac{dy}{dx}$ where $y=g(x)$ **Chain**

5. The derivative of $x=\boldsymbol{f^{-1}(y)}$ is $\dfrac{dx}{dy}=\dfrac{1}{dy/dx}$ **Inverse**

6. When $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$, what about $f(x)/g(x)$? **l'Hôpital**

$\lim \dfrac{f(x)}{g(x)} = \lim \dfrac{df/dx}{dg/dx}$ if these limits exist. Normally this is $\dfrac{\boldsymbol{f'(a)}}{\boldsymbol{g'(a)}}$

Fundamental Theorem of Calculus

If $f(x)=\displaystyle\int_a^x s(t)\,dt$ then **derivative of integral** $=\dfrac{df}{dx}=\boldsymbol{s(x)}$

If $\dfrac{df}{dx}=s(x)$ then **integral of derivative** $=\displaystyle\int_a^b s(x)\,dx=\boldsymbol{f(b)-f(a)}$

Both parts assume that $s(x)$ is a continuous function.

All Values Theorem Suppose $f(x)$ is a continuous function for $a \leqslant x \leqslant b$. Then on that interval, $f(x)$ reaches its maximum value M and its minimum m. And $f(x)$ takes all values between m and M (there are no jumps).

Mean Value Theorem If $f(x)$ has a derivative for $a \leqslant x \leqslant b$ then

$$\frac{f(b)-f(a)}{b-a} = \frac{df}{dx}(c) \quad \text{at some } c \text{ between } a \text{ and } b$$

"At some moment c, instant speed = average speed"

Taylor Series Match all the derivatives $f^{(n)} = d^n f/dx^n$ at the basepoint $x = a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \cdots$$

$$= \boldsymbol{\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\,(x-a)^n}$$

Stopping at $(x-a)^n$ leaves the error $f^{n+1}(c)\,(x-a)^{n+1}/(n+1)!$

[c is somewhere between a and x] [$n = 0$ is the Mean Value Theorem]

The Taylor series looks best around $a = 0$ $\quad f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\,x^n$

Binomial Theorem shows Pascal's triangle

$$\begin{array}{ll}
(1+x) & \mathbf{1} + \mathbf{1}x \\
(1+x)^2 & \mathbf{1} + \mathbf{2}x + \mathbf{1}x^2 \\
(1+x)^3 & \mathbf{1} + \mathbf{3}x + \mathbf{3}x^2 + \mathbf{1}x^3 \\
(1+x)^4 & \mathbf{1} + \mathbf{4}x + \mathbf{6}x^2 + \mathbf{4}x^3 + \mathbf{1}x^4
\end{array}$$

Those are just the Taylor series for $f(x) = (1+x)^p$ when $p = 1, 2, 3, 4$

$$\begin{array}{lcccc}
f^{(n)}(x) = & (1+x)^p & p(1+x)^{p-1} & p(p-1)(1+x)^{p-2} & \cdots \\
f^{(n)}(0) = & \mathbf{1} & \boldsymbol{p} & \boldsymbol{p(p-1)} & \cdots
\end{array}$$

Divide by $n!$ to find the Taylor coefficients = **Binomial coefficients**

$$\frac{1}{n!} f^{(n)}(0) = \frac{p(p-1)\cdots(p-n+1)}{n(n-1)\cdots(1)} = \frac{p!}{(p-n)!\,n!} = \boldsymbol{\binom{p}{n}}$$

The series stops at x^n when $p = n$ Infinite series for other p

$$\text{Every } (1+x)^p = 1 + px + \frac{p(p-1)}{(2)(1)}x^2 + \frac{p(p-1)(p-2)}{(3)(2)(1)}x^3 + \cdots$$